# GENERALIZATION OF WIGNER'S UNITARY-ANTIUNITARY THEOREM FOR INDEFINITE INNER PRODUCT SPACES

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*Dedicated to Mád*

Abstract. We present a generalization of Wigner's unitary-antiunitary theorem for pairs of ray transformations. As a particular case, we get a new Wigner-type theorem for non-Hermitian indefinite inner product spaces.

The classical Wigner unitary-antiunitary theorem plays a fundamental role in the foundations of quantum mechanics and it also has deep connections with the theory of projective spaces. It states that every ray transformation (see below) on a Hilbert space which preserves the transition probabilities can be lifted to a (linear) unitary or a (conjugate-linear) antiunitary operator on $H$ (see [1, 12, 13]). So, Wigner's result concerns definite inner product spaces. On the other hand, it has become quite clear by now that the indefinite inner product spaces might be even more useful for the discussion of several physical problems. For example, this is the case in relation to the divergence problem in quantum field theory, or when one wants to preserve some basic properties of the field like relativistic covariance and locality (see the introduction of [3]). This raises the need to study Wigner's theorem in the "indefinite" setting as well. Previous results in this direction were presented in [2, 3, 4]. The aim of this paper is to contribute to this study by giving a very general Wigner-type theorem which involves not one but two ray transformations and then apply it to get a generalization of Wigner's theorem for indefinite inner product spaces. The main difference which distiguishes our result from the previous ones is that we do not assume even that the indefinite inner product under consideration is Hermitian. What allows us to reach this result is that we refine our algebraic approach to Wigner's theorem which has already been proved to be fruitful in our recent papers [10, 11]. The main feature of this approach is that instead of manipulating in the underlying space, we push the problem to an operator algebra over our space and apply some classical results from pure ring theory. Hence, our method is completely different from those used previously in the papers dealing with Wigner's theorem in indefinite inner product spaces.

Let us fix the definitions and notation that we shall use throughout. In what follows, let $H$ be a Hilbert space. Given a vector $x \in H$, the set of all vectors of the form $\lambda x$ with $\lambda \in \mathbb{C}$, $|\lambda| = 1$ is called the ray associated to $x$ and it is denoted by $\underline{x}$. For any $x, y \in H$ we define

$$\underline{x} \cdot \underline{y} = |\langle x, y\rangle|.$$

The notation $\underline{H}$ stands for the set of all rays in $H$. The algebra of all bounded linear operators on $H$ is denoted by $B(H)$, and $F(H)$ stands for the ideal of all finite rank operators in $B(H)$. If $x, y \in H$ are arbitrary vectors, then $x \otimes y$ is an element of $F(H)$ which is defined by $(x \otimes y)z = \langle z, y\rangle x$ $(z \in H)$. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ between the algebras $\mathcal{A}$ and $\mathcal{B}$ is called a Jordan homomorphism if

$$\phi(x^2) = \phi(x)^2 \qquad (x \in \mathcal{A}).$$

Our main result which follows presents a Wigner-type result for pairs of ray transformations.

**Theorem.** *Let $H$ be a complex Hilbert space of dimension at least 3. Let $T, S : \underline{H} \to \underline{H}$ be bijective transformations with the property that*

$$T\underline{x} \cdot S\underline{y} = \underline{x} \cdot \underline{y} \qquad (\underline{x}, \underline{y} \in \underline{H}).$$

*Then there are bounded invertible either both linear or both conjugate-linear operators $U, V : H \to H$ such that $V = U^{*-1}$ and*

$$T\underline{x} = \underline{Ux}, \qquad S\underline{x} = \underline{Vx} \qquad (x \in H).$$

*Proof.* For every $x \in H$ pick a vector from $T\underline{x}$. In that way we get a function, which will be denoted by the same symbol $T$, from $H$ into itself with the property that for every vector $y \in H$, there exists a vector $x \in H$ such that $y = \lambda Tx$ for some $\lambda \in \mathbb{C}$ of modulus 1. Let us do the same with the other transformation $S$. Clearly, we have

$$|\langle Tx, Sy\rangle| = |\langle x, y\rangle| \qquad (x, y \in H).$$

Obviously, for every unit vector $x \in H$ we can choose a scalar $\lambda_x$ with $|\lambda_x| = 1$ such that $\lambda_x\langle Tx, Sx\rangle = 1$. By the properties of our original transformation $T$, we can clearly suppose that here in fact we have $\langle Tx, Sx\rangle = 1$. We define a function $\mu$ on the set $P_f(H)$ of all finite rank projections (self-adjoint idempotents) on $H$ as follows. If $P \in P_f(H)$, then there are pairwise orthogonal unit vectors $x_1, \ldots, x_n \in H$ such that $P = x_1 \otimes x_1 + \ldots + x_n \otimes x_n$. We set

$$\mu(P) = Tx_1 \otimes Sx_1 + \ldots + Tx_n \otimes Sx_n.$$

Apparently, the operators $Tx_1 \otimes Sx_1, \ldots, Tx_n \otimes Sx_n$ are pairwise orthogonal rank-one idempotents (two idempotents $P, Q$ are said to be orthogonal if $PQ = QP = 0$). Hence, $\mu(P)$ is a rank-$n$ idempotent. We have to check that $\mu$ is well-defined. This follows from the following observation. We have

$$\text{rng } (\sum_{k=1}^{n} Tx_k \otimes Sx_k) = [Tx_1, \ldots, Tx_n]$$

and

$$\ker\,(\sum_{k=1}^{n} Tx_k \otimes Sx_k) = [Sx_1, \dots, Sx_n]^{\perp},$$

where [.] denotes generated subspace. Now, suppose that the pairwise orthogonal unit vectors $x_1', \dots, x_n'$ generate the same subspace as $x_1, \dots, x_n$ do. Let $y \in H$. Then there exist a vector $x \in H$ and a scalar $\lambda$ of modulus 1 such that $y = \lambda Sx$. We have

$$y \perp [Tx_1, \dots, Tx_n] \Leftrightarrow Sx \perp [Tx_1, \dots, Tx_n] \Leftrightarrow$$

$$x \perp [x_1, \dots, x_n] \Leftrightarrow x \perp [x_1', \dots, x_n'] \Leftrightarrow \tag{1}$$

$$Sx \perp [Tx_1', \dots, Tx_n'] \Leftrightarrow y \perp [Tx_1', \dots, Tx_n'].$$

This shows that the range of $\sum_{k=1}^{n} Tx_k \otimes Sx_k$ is the same as that of $\sum_{k=1}^{n} Tx_k' \otimes Sx_k'$. The same applies for the kernels. Since the idempotents are determined by their ranges and kernels, this proves that $\mu$ is well-defined. It is now clear that $\mu$ is an orthoadditive measure on $P_f(H)$. We show that $\mu$ is bounded on the set $P_1(H)$ of all rank-one projections which is equivalent to

$$\sup_{\|x\|=1} \|Tx\|\|Sx\| < \infty.$$

Suppose, on the contrary, that there is a sequence $(u_n)$ of unit vectors in $H$ for which $\|Tu_n\|\|Su_n\| \longrightarrow \infty$. Since $(u_n)$ is bounded, it has a subsequence $(u_{k_n})$ weakly converging to a vector, say, $u \in H$. We have

$$|\langle Tu_{k_n}, Sv\rangle| = |\langle u_{k_n}, v\rangle| \longrightarrow |\langle u, v\rangle|.$$

Since this holds for every $v \in H$, we deduce that $(Tu_{k_n})$ is weakly bounded which implies that it is in fact norm-bounded. The same argument applies in relation to $S$. Hence, we obtain that $(u_n)$ has a subsequence $(u_{l_n})$ such that $\|Tu_{l_n}\|, \|Su_{l_n}\|$ are bounded which is a contradiction. Consequently, $\mu$ is bounded on $P_1(H)$.

By Gleason's theorem $\mu$ can be extended to a Jordan homomorphism of $F(H)$. In fact, if $A \in F(H)$ is self-adjoint, then there are finite-rank projections $P_1, \dots, P_n$ (here, we do not require that they are pairwise orthogonal) and scalars $\lambda_1, \dots, \lambda_n$ such that $A = \lambda_1 P_1 + \dots + \lambda_n P_n$. Let

$$\phi(A) = \lambda_1 \mu(P_1) + \dots + \lambda_n \mu(P_n).$$

Consider a finite dimensional subspace $H_0$ of $H$ with dimension at least 3 which contains all the subspaces $\operatorname{rng} A, \ker A^{\perp}, \operatorname{rng} P_1, \dots, \operatorname{rng} P_n$. Since $\mu$ is bounded on $P_1(H_0)$, by the variation [5, Theorem 3.2.16] of Gleason's theorem, for every $x, y \in H$ there is an operator $T_{xy}$ on $H_0$ such that

$$\langle \lambda_1 \mu(P_1) + \dots + \lambda_n \mu(P_n) x, y\rangle = \lambda_1 \langle \mu(P_1)x, y\rangle + \dots + \lambda_n \langle \mu(P_n)x, y\rangle =$$

$$\lambda_1 \operatorname{tr}\,(P_1 T_{xy}) + \dots + \lambda_n \operatorname{tr}\,(P_n T_{xy}) = \operatorname{tr}\,(A T_{xy}).$$

We now easily obtain that $\phi$ is well-defined and real-linear on the set of all self-adjoint finite rank operators. If $A \in F(H)$ is arbitrary, then there exist self-adjoint finite rank operators $A_1, A_2$ such that $A = A_1 + iA_2$. Define $\phi(A) = \phi(A_1) + i\phi(A_2)$. Clearly, $\phi$ is a linear map on $F(H)$ which sends projections to idempotents. It is a standard algebraic argument to verify that $\phi$ is then a Jordan homomorphism (see, for example, the proof of [9, Theorem 2]). Since $F(H)$ is a locally matrix ring, we can apply a classical theorem of Jacobson and Rickart. By [8, Theorem 8] we obtain that $\phi$ can be written as $\phi = \phi_1 + \phi_2$, where $\phi_1$ is a homomorphism and $\phi_2$ is an antihomomorphism. Since $\phi(P)$ is a rank-one idempotent and $\phi_1(P)$, $\phi_2(P)$ are idempotents, we infer from $\phi(P) = \phi_1(P) + \phi_2(P)$ that either $\phi_1(P) = 0$ or $\phi_2(P) = 0$. Since the ring $F(H)$ is simple, we obtain that either $\phi_1 = 0$ or $\phi_2 = 0$. Therefore, $\phi$ is either a homomorphism or an antihomomorphism. Without loss of generality we can assume that $\phi$ is a homomorphism. We assert that $\phi$ is rank-preserving. Let $A \in F(H)$ be a rank-$n$ operator. Then there is a rank-$n$ projection $P$ such that $PA = A$. The rank of $\phi(P)$ is also $n$. We have $\phi(A) = \phi(PA) = \phi(P)\phi(A)$ which proves that $\phi(A)$ is of rank at most $n$. If $Q$ is any rank-$n$ projection, then there are finite rank operators $U, V$ such that $Q = UAV$. Since $\phi(Q) = \phi(U)\phi(A)\phi(V)$ and the rank of $\phi(Q)$ is $n$, it follows that the rank of $\phi(A)$ is at least $n$. Therefore, $\phi$ is rank-preserving. We now refer to Hou's work [6] on the form of linear rank preservers on operator algebras. It follows from the argument leading to [6, Theorem 1.2] that there are linear operators $U, V$ on $H$ such that $\phi$ is of the form

$$\phi(x \otimes y) = (Ux) \otimes (Vy) \qquad (x, y \in H) \tag{2}$$

(recall that we have assumed that $\phi$ is a homomorphism). If $x \in H$ is a unit vector, then we have $Tx \otimes Sx = \phi(x \otimes x) = Ux \otimes Vx$. Taking traces, we obtain $1 = \langle Tx, Sx \rangle = \langle Ux, Vx \rangle$. Since this holds for every unit vector $x$, by the linearity of $U, V$, using polarization we get that

$$\langle Ux, Vy \rangle = \langle x, y \rangle \qquad (x, y \in H). \tag{3}$$

We assert that $U, V$ are surjective. Consider, for example, the case of $U$. Let $0 \neq x \in H$ be any vector and let $0 \neq \lambda \in \mathbb{C}$ be any scalar. It is easy to see that $[Tx]^\perp = [T(\lambda x)]^\perp$ (see (1)). Therefore, $T(\lambda x) = \lambda' Tx$ with some scalar $\lambda'$. Denote $x_e = x/\|x\|$. We compute

$$Ux \otimes Vx = \|x\|^2 Ux_e \otimes Vx_e = \|x\|^2 \phi(x_e \otimes x_e) = \|x\|^2 Tx_e \otimes Sx_e.$$

This gives us that $Tx_e \in [Ux]$. But $Tx$ is in the one-dimensional subspace generated by $Tx_e$. So, we have

$$Tx \in [Ux]. \tag{4}$$

Since rng $U$ is a linear subspace of $H$ and $T$ is "almost" surjective, we obtain the surjectivity of $U$. Similar argument applies to $V$. We next show that $U, V$ are bounded. Let $(x_n)$ be a sequence converging to 0 and let $y \in H$ be

such that $Ux_n \to y$. If $x \in H$ is arbitrary, then we have

$$\langle Ux_n, Vx\rangle = \langle x_n, x\rangle \longrightarrow 0.$$

Since $V$ is surjective, we obtain that $(Ux_n)$ weakly converges to 0. It follows that $y = 0$. By the closed graph theorem we deduce that $U$ is bounded. Similar argument proves the boundedness of $V$. It follows from (3) that $V^*U = I$. This gives us that $U$ is injective. Therefore, $U$ and $V$ are invertible and $V = U^{*-1}$.

By (4) and the similar relation $Sx \in [Vx]$ $(x \in H)$, there are functions $\varphi, \psi : H \to \mathbb{C}$ such that

$$Tx = \varphi(x)Ux, \qquad Sx = \psi(x)Vx \qquad (x \in H).$$

We have

$$|\varphi(x)||\psi(y)||\langle x, y\rangle| = |\varphi(x)||\psi(y)||\langle Ux, Vy\rangle| = |\langle Tx, Sy\rangle| = |\langle x, y\rangle|,$$

that is, $|\varphi(x)||\psi(y)| = 1$ if $\langle x, y\rangle \neq 0$. This easily implies that $|\varphi|$ and $|\psi|$ are both constant. Multiplying $U, V, \varphi, \psi$ by suitable constants, we obtain the statement of the theorem. The proof is complete. $\square$

In the following corollary of our theorem we give a generalization of Wigner's theorem for the indefinite inner product space generated by any invertible operator $A \in B(H)$. Since we do not assume that $A$ is self-adjoint, this result can, in some sense, be considered as a generalization of the results in [2, 3].

**Corollary 1.** *Let $H$ be a complex Hilbert space with $\dim H \geq 3$ and let $A \in B(H)$ be invertible. For any $x, y \in H$ define $\underline{x} \cdot_A \underline{y} = |\langle Ax, y\rangle|$. Let $T : \underline{H} \to \underline{H}$ be a bijective transformation such that*

$$T\underline{x} \cdot_A T\underline{y} = \underline{x} \cdot_A \underline{y} \qquad (x, y \in H).$$

*Then there is a bounded invertible either linear or conjugate-linear operator $U$ on $H$ with $U^*AU = \epsilon A$ for some scalar $\epsilon$ of modulus 1 such that*

$$T\underline{x} = \underline{Ux} \qquad (x \in H).$$

*Proof.* Just as in the proof of our theorem above, we can define an "almost" surjective map (that is, which has values in every ray) on the underlying Hilbert space $H$ denoted by the same symbol $T$ such that

$$|\langle ATx, Ty\rangle| = |\langle Ax, y\rangle| \qquad (x, y \in H).$$

Set $S = ATA^{-1}$. The proof of our theorem now applies and we find that there is a bounded invertible either linear or conjugate-linear operator $U$ on $H$ and a scalar function $\varphi : H \to \mathbb{C}$ such that $Tx = \varphi(x)Ux$ $(x \in H)$. Since

$$|\varphi(x)||\varphi(y)||\langle AUx, Uy\rangle| = |\langle ATx, Ty\rangle| = |\langle Ax, y\rangle| \qquad (x, y \in H), \tag{5}$$

it follows that $[U^*AUx]^\perp = [Ax]^\perp$ for every $x \in H$. Therefore, the linear operators $U^*AU$ and $A$ are locally linearly dependent which means that $U^*AUx$ and $Ax$ are linearly dependent for every $x \in H$. Since none of the operators $U^*AU$ and $A$ is of rank 1, by [7, Lemma 3] we obtain that there is

a scalar $c$ such that $U^*AU = cA$. Let $x, y \in H$ be arbitrary nonzero vectors. Pick $z \in H$ such that $\langle Ax, z\rangle, \langle Ay, z\rangle \neq 0$. From (5) we now infer that

$$|\varphi(x)||\varphi(z)||c| = 1, \qquad |\varphi(y)||\varphi(z)||c| = 1.$$

This shows that $|\varphi|$ is constant. If $d$ denotes this constant, then we have $d^2|c| = 1$. Let $\epsilon = d^2 c$. Then $\epsilon$ is of modulus 1 and we have

$$(d\epsilon U)^* A (d\epsilon U) = d^2 U^* AU = d^2 cA = \epsilon A.$$

Consider the factorization

$$Tx = \left(\frac{1}{d\epsilon}\varphi(x)\right)(d\epsilon U).$$

Since $\frac{1}{d\epsilon}\varphi(x)$ is of modulus 1, the proof is complete. $\square$

In the finite dimensional case, Corollary 1 can be reformulated in the following way.

**Corollary 2.** *Let $H$ be a finite dimensional complex Hilbert space with $\dim H \geq 3$. Let $B : H \times H \to \mathbb{C}$ be a sesquiliner form which is non-degenerate in the sense that $B(x, y) = 0$ $(y \in H)$ implies $x = 0$. Define $\underline{x} \cdot_B \underline{y} = |B(x, y)|$ $(x, y \in H)$. Let $T : \underline{H} \to \underline{H}$ be a bijective transformation such that*

$$T\underline{x} \cdot_B T\underline{y} = \underline{x} \cdot_B \underline{y} \qquad (x, y \in H).$$

*Then either there is an invertible linear operator $U$ on $H$ such that $B(Ux, Uy) = \epsilon B(x, y)$ $(x, y \in H)$ for some scalar $\epsilon$ of modulus 1 and*

$$T\underline{x} = \underline{Ux} \qquad (x \in H),$$

*or there is an invertible conjugate-linear operator $U'$ on $H$ such that $\overline{B(U'x, U'y)} = \epsilon' B(x, y)$ $(x, y \in H)$ for some scalar $\epsilon'$ of modulus 1 and*

$$T\underline{x} = \underline{U'x} \qquad (x \in H).$$

*Proof.* Since $H$ is finite dimensional, it is easy to see that there exists an invertible linear operator $A$ on $H$ such that $B(x, y) = \langle Ax, y\rangle$ $(x, y \in H)$. Now, Corollary 1 applies. $\square$

*Remark* 1. Our results are valid in real Hilbert spaces as well. In order to see it, we must refine the argument we have presented in the complex case. Namely, one can follow the argument that has been applied in the proof of [10, Theorem 3]. Observe that in the papers [2, 3] the authors considered only complex spaces.

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## References


[1] Bargmann, V.: Note on Wigner's theorem on symmetry operations. J. Math. Phys. **5**, 862–868 (1964)

[2] Baum, M.: The Wigner-Bargmann theorem in the pseudo-unitary case. Ann. Global Anal. Geom. **1**, 119–128 (1983)

[3] Bracci, L., Morchio, G., Strocchi, F.: Wigner's theorem on symmetries in indefinite metric spaces. Commun. Math. Phys. **41**, 289–299 (1975)

[4] van den Broek, P.M.: Symmetry transformations in indefinite metric spaces. Phys. A. **127**, 599–612 (1984)

[5] Dvurečenskij, A.: *Gleason's Theorem and Its Applications.* Dordrecht: Kluwer Academic Publishers, 1993

[6] Hou, J.C.: Rank-preserving linear maps on $B(X)$. Sci. China Ser. A **32**, 929–940 (1989)

[7] Hou, J.C.: Linear interpolation and elementary operators on $B(X)$. Sci. China Ser. A **36**, 1025–1035 (1993)

[8] Jacobson, N., Rickart, C.: Jordan homomorphisms of rings. Trans. Amer. Math. Soc. **69**, 479–502 (1950)

[9] Molnár, L.: The set of automorphisms of $B(H)$ is topologically reflexive in $B(B(H))$. Studia Math. **122**, 183–193 (1997)

[10] Molnár, L.: An algebraic approach to Wigner's unitary-antiunitary theorem. J. Austral. Math. Soc. **65**, 354–369 (1998)

[11] Molnár, L.: A generalization of Wigner's unitary-antiunitary theorem to Hilbert modules. J. Math. Phys. **40**, 5544–5554 (1999)

[12] Rätz, J.: On Wigner's theorem: Remarks, complements, comments, and corollaries. Aequationes Math. **52**, 1–9 (1996)

[13] Sharma, C.S., Almeida, D.F.: A direct proof of Wigner's theorem on maps which preserve transition probabilities between pure states of quantum systems. Ann. Phys. **197**, 300–309 (1990)



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